

Math 261B Thurs. 9/24

$SL_2(k)$ char $k = 0$

Finite-dim reps are \oplus 's of irreducibles
and irreducibles = standard reps

$$Sd = k[x, y]_d$$

In all characteristics:

- local finiteness of comodules, all irr. reps appear in $\mathcal{O}(G)$,
affine \Rightarrow linear

- Unipotent groups, $\mathcal{Z}_u(G)$,

\Downarrow
only irr. rep. is trivial \iff all fin dim reps. $G \rightarrow GL_n$
factor through $U =$ upper unitriangular
 $U \in GL_n$
in some basis

- Solvable groups, $\mathcal{Z}(G)$

\Downarrow all irr. rep. are 1 dimensional

- tori \mathbb{G}_m^n , complete reducibility characters & cocharacter

- definition: G is reductive if $R_u(G) = 1$. (char 0 \Leftrightarrow complete reducibility)
- Reductive G classified by root data $(X, X^*, R, R^\vee) \dots$

- Irreducible reps. of reductive G are classified by dominant weights

$$V_\lambda: \lambda \in X \text{ s.t. } \langle \alpha^\vee, \lambda \rangle \geq 0 \text{ for } \alpha \in R_+$$

V_λ characterized by having a U -invariant vector of weight λ
(highest weight vector)

$$\begin{aligned} G &= B = T \\ B &= T \rtimes U \\ U &= R_u(B) \end{aligned}$$

Different in char $\neq 0$:

- Reps aren't completely reducible
- $V_\lambda \not\cong S_\lambda \leftarrow$ "standard" rep. with HW λ .

Ex. SL_2 $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $ad - bc - 1 = 0$ x^n, x^{n-1}, y, \dots

acts on $v(x, y)$ $f(x, y) \rightarrow f(ax+cy, bx+dy)$

$S_d = k[x, y]_d$ $\mathbb{Z}[x, y]_d$ $\mathbb{Z}[a, b, c, d] / (\det - 1)$

x_{ij}

$\det(x) - 1 = 0$

$(XY)_{ij} = \sum_k x_{ik} y_{kj}$

$\Delta x_{ij} = \sum_k x_{ik} \otimes x_{kj}$

$\mathbb{Z}[x_{ij}] / (\det(x) - 1)$

$k \otimes_{\mathbb{Z}} - : \mathcal{O}(SL_n)$

$SL_2(k)$

$SL_2(k) \rightsquigarrow S_d$

$$\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}$$

$$t \mapsto t^a$$

$a \in \mathbb{Z}$

Re-examine irreducible reps. of SL_2

$$X = \mathbb{Z}$$

Standard reps $k[x, y]$: \uparrow basis weight

$$x^d, x^{d-1}y, \dots, y^d$$

$$\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} f(x, y) \mapsto f(tx, t^{-1}y)$$

$$\begin{matrix} \uparrow & & \uparrow & & \uparrow \\ -t^d & & t^{d-1} \cdot t^{-1} & & \dots & & t^{-d} \end{matrix}$$

weights $d, d-2, d-4, \dots, -d$

$$\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} f(x, y) \mapsto f(x, y+bx) = f(x, y) \text{ identically in } b$$

$$f = c x^d$$

$V_d = S_d$ is submodule gen. by x^d .

$SL_2(\overline{\mathbb{F}}_3)$

$$V_0 = S_0$$

$$V_1 = S_1$$

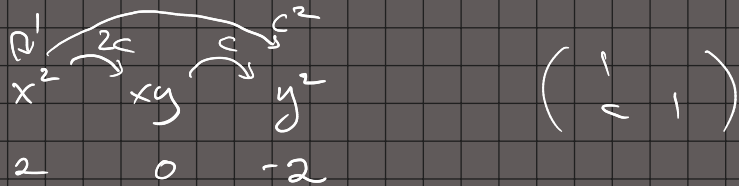
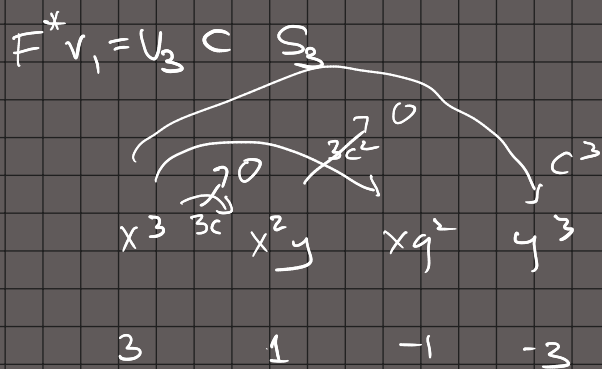
$$V_2 = S_2$$

$$\begin{matrix} 1 \\ x & y \\ 1 & -1 \end{matrix}$$

$$\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \\ c & 1 \end{pmatrix}$$

$$\begin{matrix} x & \xleftarrow{b} & y \\ & & \uparrow \\ x & \xrightarrow{c} & y \end{matrix}$$



$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^p & b^p \\ c^p & d^p \end{pmatrix}$$

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \mapsto \left(\det \begin{pmatrix} a^p & b^p \\ c^p & d^p \end{pmatrix} = 1 \right)^p$$

is a homeomorphism
 $F: SL_2(k) \rightarrow SL_2(k)$ $\left(\det \begin{pmatrix} a^p & b^p \\ c^p & d^p \end{pmatrix} = 1 \right)$

$$x^3 \mapsto (x+cy)^3$$

$$\begin{pmatrix} 1 & \\ & c \end{pmatrix}^p$$

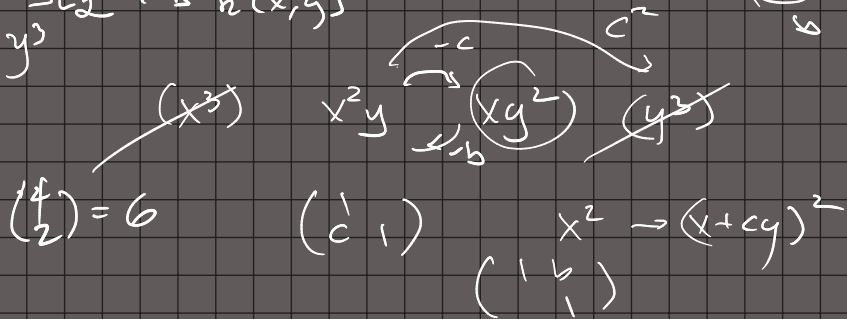
$$x^3 + \binom{3}{1} x^2y + \binom{3}{2} xy^2 + y^3$$

$SL_2 \xrightarrow{\sim} k[x,y]$
 $\downarrow F \quad F \downarrow \quad x, y \mapsto x^p, y^p \quad p=3$

$SL_2 \xrightarrow{\sim} k[x,y]$

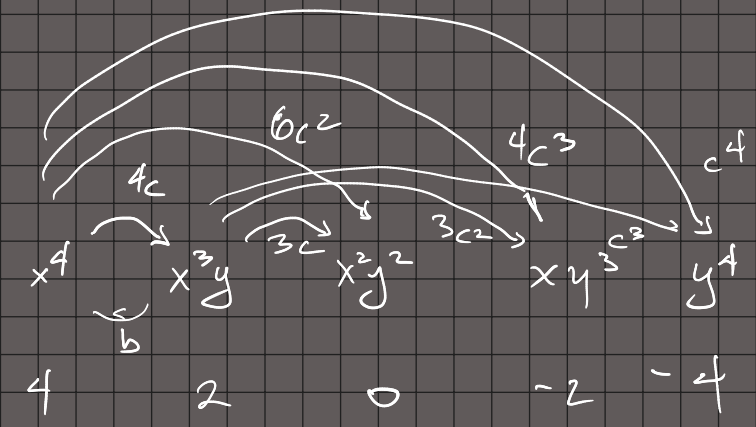
$$0 \rightarrow V_3 \rightarrow S_3 \rightarrow V_1 \rightarrow 0$$

\parallel
 $F^* V_1$



$$V_1 \otimes V_3 \cong V_4 \oplus V_0$$

$$0 \rightarrow V_4 \rightarrow S_4 \rightarrow V_0 \rightarrow 0$$



$$(x+cy)^3$$

$$S_1 \otimes S_3 \rightarrow S_4$$

$$\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ c & 1 \end{pmatrix} (x+cy)^4$$

$$\langle x^4, x^3y, xy^3, y^4 \rangle = V_4$$

$$\begin{pmatrix} 0 & -v \\ v^{-1} & 0 \end{pmatrix} \text{ and } \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}$$

normalize T in SL_2
 $N(T)/T \cong S_2$

$$V_3 \begin{pmatrix} x^3 + x^{-3} \end{pmatrix} \quad V_1 \begin{pmatrix} x + x^{-1} \end{pmatrix}$$

$$(x^3 + x^{-3})(x + x^{-1}) = x^4 + x^2 + x^{-2} + x^{-4}$$

Val

acts on t by $t \mapsto t^{-1}$

$$\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \mapsto \begin{pmatrix} t^{-1} & \\ & t \end{pmatrix}$$

as $m \mapsto -m$ on weights

In $SL_n / GL_n / PGL_n$

$$\begin{pmatrix} * & & & \\ & * & & \\ * & & \dots & \\ & * & & \end{pmatrix} \in \mathbb{Z}^n$$

In reductive group G :

$$\mathfrak{g} = \underbrace{\mathfrak{t}}_{\text{Lie}(\mathcal{T})} \oplus \bigoplus_{\alpha \in \mathcal{R}} \mathfrak{g}_\alpha$$

$$N(\mathcal{T}) / \mathcal{T} = S_n$$

$\overset{W}{\curvearrowright}$
acts on \mathcal{T}

each $\mathfrak{g}_\alpha \oplus \mathfrak{k}_\alpha \oplus \mathfrak{g}_\alpha$ is image of $\text{Lie}(SL_2)$ for $SL_2 \xrightarrow{\alpha} G$

$$sl_2 = \mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2$$

$\mathfrak{g}_\alpha \quad \mathfrak{t} \quad \mathfrak{g}_\alpha$

$$\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}$$

\cong

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$t^{-2} \quad 1 \quad t^2$

\mathbb{C}^*

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \xrightarrow{\alpha} U_\alpha \subset G$$

\mathcal{T} and $U_\alpha \subset G$

generate $\mathcal{T} \times U_\alpha$ with

group law

$$\begin{pmatrix} a & \\ & t \end{pmatrix} \cdot \begin{pmatrix} b & \\ & s \end{pmatrix}$$

\mathbb{C}^*
 $\cong \mathbb{C}^*$

$$\begin{pmatrix} t & b t^{-1} \\ & t \end{pmatrix}$$

\mathbb{C}^*
 $\cong \mathbb{C}^*$

$$(a + t^\alpha b, t \cdot \underline{c})$$

$$\mathcal{O}(\mathbb{T} \times U_d) = k[t_1^{\pm 1}, \dots, t_r^{\pm 1}, a]$$

$\mathbb{G}_m^r \times \mathbb{G}_a$

$$G \cong V \quad V = \bigoplus_{\lambda \in X} V_\lambda \text{ as } \mathbb{T} \text{ module.}$$

$$\Delta t_i = t_i \otimes t_i$$

$$\Delta a = a \otimes 1 + t^\alpha \otimes a$$

$$\mathbb{T} \times U_d \cong V \quad v \in V_\lambda$$

$$\text{Claim: } (\mathbb{T} \times U_d) \cdot v \subset V_\lambda + V_{\lambda+\alpha} + V_{\lambda+2\alpha} + \dots$$

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