Math 261B Thurs. 9/24

$SL_2(k)$ char $k=0$ Finite-dim'l reps are $\oplus$'s of irreducibles and irreducibles = standard reps

$$S_d = k[x,y]_d$$

In all characteristics:

- local finiteness of comodules, all irr. reps appear in $\mathcal{O}(G)$, affine ⇒ linear
- Unipotent groups, $R_u(G)$,

  $\Updownarrow$

  only irr. rep. is trivial $\Longleftrightarrow$ all fin dim reps. $G \to GL_n$ factor through U = upper unitriangular $U \subseteq GL_n$ in some basis
- Solvable groups, $R(G)$

  $\Downarrow$

  all irr. rep. are 1 dimensional
- tori $\mathbb{G}_m^n$, complete reducibility characters & cocharacter

- definition: G is reductive if $R_u(G) = 1$. (char 0 $\Leftrightarrow$ complete reducibility)
- Reductive G classified by root data $(X, X^*, R, R^\vee)$ ...
- Irreducible reps. of reductive G are classified by dominant weights

  V_λ : $\lambda \in X$ s.t. $\langle \alpha^\vee, \lambda \rangle \geq 0$ for $\alpha \in R_+$

  V_λ characterized by having a U-invariant vector of weight λ (highest weight vector)

  $G \supset B \supset T$

  $B = T \ltimes U$

  $U = R_u(B)$

Different in char $\neq 0$:

- Reps aren't completely reducible
- $V_\lambda \neq S_\lambda$ a "standard" rep. with HW λ.

$\boxed{x_{ij}}$

$\boxed{\det(x) - 1} = 0$

$(XY)_{ij} = \sum X_{ik} Y_{kj}$

$\Delta x_{ij} = \sum x_{ik} \otimes x_{kj}$

$\mathbb{Z}[x_{ij}]/(\det(x) - 1)$

$K \otimes_{\mathbb{Z}} -$: $\mathcal{O}(SL_n)$

Ex. SL_2 $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $ad - bc - 1 = 0$

$x^n, x^{n-1}, y, \dots$

acts on $k[x,y]$ $f(x,y) \to \boxed{f(ax+cy, bx+dy)}$

$S_d = k[x,y]_d$ $\mathbb{Z}[x,y]_d$

$\mathbb{Z}[a,b,c,d]/(\det - 1)$

$SL_2(K)$ $\quad SL_2(K) \curvearrowright S_d$ $\quad \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}$ $\quad t \mapsto t^a$

$a \in \mathbb{Z}$

$X = \mathbb{Z}$

Re-examine irreducible reps. of SL_2

Standard reps $K[x,y]_d$: weight basis

$$\begin{array}{ccccc} x^d & , & x^{d-1}y, & \ldots, & y^d \\ \uparrow & & \uparrow & & \uparrow \\ \cdot t^d & & t^{d-1}\cdot t^{-1} & \ldots & t^{-d} \end{array}$$

$\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} f(x,y) \rightsquigarrow f(tx, t^{-1}y)$

Weights $d, \quad d-2, \quad d-4, \ldots, -d$

$\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} f(x,y) \mapsto f(x, y+bx) \cdot= f(x,y)$ identically in b

$\Updownarrow$

$f = c x^d$ $\qquad V_d \subset S_d$ is submodule gen. by x^d.

$SL_2(\mathbb{F}_3)$

$V_0 = S_0 \qquad 1$

$V_1 = S_1 \qquad \underset{1}{x} \quad \underset{-1}{y}$

$V_2 = S_2$

$\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}$
$$\begin{tikzcd} x & y \arrow[l, bend right, "b"'] \arrow[loop above, "1"] \end{tikzcd}$$

$\begin{pmatrix} 1 & \\ c & 1 \end{pmatrix}$
$$\begin{tikzcd} x \arrow[loop above, "1"] \arrow[r, bend right, "c"'] & y \end{tikzcd}$$

$F^*V_1 = V_3 \subset S_3$

$$\begin{tikzcd} x^3 \arrow[r, bend left, "3c \to 0"] \arrow[rr, bend left, "3c^2 \to 0"] \arrow[rrr, bend left, "c^3"] & x^2y & xy^2 & y^3 \\ 3 & 1 & -1 & -3 \end{tikzcd}$$

$x^3 \mapsto (x+cy)^3$

$\begin{pmatrix} 1 & \\ c & 1 \end{pmatrix}$

$\parallel$

$x^3 + \underset{3}{\binom{3}{1}} x^2y + \underset{3}{\binom{3}{2}} xy^2 + y^3$

$$\begin{tikzcd} 0 \arrow[r] & V_3 \arrow[r] \arrow[d, equal] & S_3 \arrow[r] & V_1 \arrow[r] & 0 \\ & F^*V_1 & & & \end{tikzcd}$$

$$\begin{tikzcd} x^2 \arrow[r, bend left, "2c"] \arrow[rr, bend left, "c^2"] & xy \arrow[r, bend left, "c"] & y^2 \\ 2 & 0 & -2 \end{tikzcd}$$

$SL_{2F} \to SL_2 \curvearrowright V_1$

$\begin{pmatrix} 1 & \\ c & 1 \end{pmatrix}$

$x \mapsto x + cy$

$x^2 + 2cxy + c^2y^2$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^p & b^p \\ c^p & d^p \end{pmatrix}$

$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} - 1 \mapsto \left(\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} - 1\right)^p$

$\rotatebox{90}{$\in$}$

$\left(\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} - 1\right)$

is a homomorphism

$F : SL_2(k) \to SL_2(k)$

$$\begin{tikzcd} SL_2 \arrow[r, phantom, "\curvearrowright"] \arrow[d, "F"] & k[x,y] \arrow[d, "F"'] \\ SL_2 \arrow[r, phantom, "\curvearrowright"] & k[x,y] \end{tikzcd}$$

$x, y \mapsto x^p, y^p$

$p = 3$

$$\begin{tikzcd} \bullet \arrow[r, bend left, "c"] & \bullet \arrow[l, bend left, "b"] \end{tikzcd}$$

$$\begin{tikzcd} \cancel{(x^3)} & x^2y \arrow[r, bend left, "-c"] \arrow[rr, bend left, "c^2"] & (xy^2) \arrow[l, bend left, "b"] & \cancel{(y^3)} \end{tikzcd}$$

$\binom{4}{2} = 6$

$\begin{pmatrix} 1 & \\ c & 1 \end{pmatrix}$

$x^2 \to (x+cy)^2$

$\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}$

$V_1 \otimes V_3$

$\parallel$

$0 \to V_4 \to S_4 \to V_0 \to 0$

$$\begin{tikzcd}
x^4 \arrow[r, bend left, "4c"] \arrow[rr, bend left, "6c^2"] \arrow[rrr, bend left, "4c^3"] \arrow[rrrr, bend left, "c^4"] & x^3y \arrow[l, bend left, "b"] \arrow[r, bend left, "3c"] \arrow[rr, bend left, "3c^2"] \arrow[rrr, bend left, "c^3"] & x^2y^2 & xy^3 & y^4
\end{tikzcd}$$

$$\begin{array}{c} x^3 \\ \downarrow \\ (x+cy)^3 \end{array}$$

$4 \quad 2 \quad 0 \quad -2 \quad -4$

$S_1 \otimes S_3 \longrightarrow S_4$

$\cup$

V_3

$\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \quad \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & \\ c & 1 \end{pmatrix} \quad (x+cy)^4$

$\langle x^4,\ x^3y,\ \quad xy^3,\ y^4 \rangle = V_4$

$\boxed{\begin{pmatrix} 0 & -v \\ v^{-1} & 0 \end{pmatrix}}$ and $\overset{T}{\overset{\parallel}{\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}}}$

normalize T in SL_2

$N(T) / T \cong S_2$

$3, -3 \qquad 1, -1$

$4, 2, -2, -4$

$V_3 \quad (x^3 + x^{-3}) \qquad V_1 \quad (x + x^{-1})$

$$(x^3+x^{-3})(x+x^{-1})$$

$$= \underbrace{x^4 + x^2 + x^{-2} + x^{-4}}_{V_4}$$

acts on $\mathfrak{t}$ by $t \mapsto t^{-1}$

$$\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \mapsto \begin{pmatrix} t^{-1} & \\ & t \end{pmatrix}$$

as $m \mapsto -m$ on weights

In $SL_n / GL_n / PGL_n$

$$\begin{pmatrix} * & & \\ & & * \\ * & & \cdots \\ & * & \end{pmatrix} \qquad \mathbb{Z}^n$$

$$\underset{\underset{W}{\cap}}{N(T)/T} = S_n$$

acts on T

In reductive group G:

$$\mathfrak{g} = \underset{\underset{\mathrm{Lie}(T)}{\parallel}}{\mathfrak{t}} \oplus \bigoplus_{\alpha\in R} \mathfrak{g}_\alpha$$

each $\mathfrak{g}_{-\alpha} \oplus \underset{\underset{\mathfrak{t}}{\cap}}{k h_\alpha} \oplus \mathfrak{g}_\alpha$ is image of $\mathrm{Lie}(SL_2)$ for

$$\begin{tikzcd} SL_2 \arrow[r, "i_\alpha"] & G \\ \mathbb{G}_m \arrow[u, phantom, "\cup"] \arrow[r, "\alpha^\vee"] & T \end{tikzcd}$$

$$sl_2 = \mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2$$

$$\begin{array}{cccc} & \mathfrak{g}_{-\alpha} & \mathfrak{t} & \mathfrak{g}_\alpha \\ \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} & \begin{pmatrix} 0 & \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \wr\wr & t^{-2} & 1 & t^{2} \\ \mathbb{G}_m & & & \end{array}$$

$$\underset{\mathbb{G}_a}{\boxed{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}}} \longmapsto U_\alpha \subset G$$

T and $U_\alpha \subset G$

generate $T \ltimes U_\alpha$ with

group law

$$(\underset{\underset{\underset{\mathbb{G}_a}{\wr\wr}}{U_\alpha}}{\overset{\in}{a}}, \underset{\underset{\underset{\mathbb{G}_m^r}{\wr\wr}}{T}}{\overset{\in}{\underline{t}}}) \cdot (\underline{b}, \underline{s})$$

$$\underline{t}\, b\, \underline{t}^{-1} = \boxed{t^\alpha} b$$

$(a + t^\alpha b,\ \underline{t} \cdot \underline{s})$

$$\mathcal{O}(\underset{\substack{\rotatebox{90}{$\cong$} \\ \mathbb{G}_m^r \times \mathbb{G}_a}}{T \ltimes U_\alpha}) = k[t_1^{\pm 1}, \dots, t_r^{\pm 1}, a]$$

$G \curvearrowright V$ $\quad V = \bigoplus_{\lambda \in X} V_\lambda$ as T module.

$$\Delta t_i = t_i \otimes t_i$$

$$\Delta a = a \otimes 1 + t^\alpha \otimes a$$

$T \ltimes U_\alpha \curvearrowright V$ $\quad v \in V_\lambda$

Claim: $(T \ltimes U_\alpha) \cdot v \underset{\uparrow}{\subset} V_\lambda + V_{\lambda+\alpha} + V_{\lambda+2\alpha} + \cdots$